Sequences

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Let $a_0 = \frac{1}{2}$ and $a_{k+1} = a_k + \frac{a_k^2}{n}$. Prove that $1 - \frac{1}{n} < a_n < 1$.

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Note that $a_{k+1} = a_k + \frac{a_k^2}{n} \Leftrightarrow a_{k+1} = \frac{a_k(a_k+n)}{n} \Leftrightarrow \frac{1}{a_{k+1}} = \frac{n}{a_k(a_k+n)} \Leftrightarrow$

$\frac{1}{a_{k+1}} = \frac{1}{a_k} - \frac{1}{n+a_k} \Leftrightarrow \frac{1}{a_k} - \frac{1}{a_{k+1}} = \frac{1}{n+a_k}, k \in \mathbb{N} \cup \{0\}$.

Also note that $a_{k+1} > a_k$ for any $k \in \mathbb{N} \cup \{0\}$ (since $a_{k+1} - a_k = \frac{a_k^2}{n} > 0$).

Hence, $\sum_{k=0}^{n-1}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) = \sum_{k=0}^{n-1}\frac{1}{n+a_k} \Leftrightarrow 2 - \frac{1}{a_n} = \sum_{k=0}^{n-1}\frac{1}{n+a_k}$ and since

$a_k \geq 1/2$ for any $k \in \mathbb{N} \cup \{0\}$ then $2 - \frac{1}{a_n} \leq \sum_{k=0}^{n-1}\frac{1}{n+1/2} = \frac{n}{n+1/2} = \frac{2n}{2n+1}$.

Therefore, $2 - \frac{2n}{2n+1} \leq \frac{1}{a_n} \Leftrightarrow a_n \leq \frac{2n+1}{2n+2} < 1$.

Since $a_k < a_n, k = 0, 1, \dots, n-1$ and $a_n < 1$ we obtain that

$2 - \frac{1}{a_n} = \sum_{k=0}^{n-1}\frac{1}{n+a_k} \geq \sum_{k=0}^{n-1}\frac{1}{n+a_n} = \frac{n}{n+a_n} > \frac{n}{n+1}$. .

Hence, $\frac{1}{a_n} < 2 - \frac{n}{n+1} = \frac{n+2}{n+1} \Leftrightarrow a_n > \frac{n+1}{n+2} = 1 - \frac{1}{n+2}$ and, therefore,

$a_n - \left(1 - \frac{1}{n}\right) > 1 - \frac{1}{n+2} - \left(1 - \frac{1}{n}\right) = \frac{2}{n(n+2)} > 0$

Remark.

Using inequality $a_n \leq \frac{2n+1}{2n+2}$ we can obtain more precise lover bound for a_n,

namely $1 - \frac{1}{n+2} < a_n$. Indeed, since $a_k < a_n, k = 0, 1, \dots, n-1$ and $a_n \leq \frac{2n+1}{2n+2}$

we obtain $2 - \frac{1}{a_n} = \sum_{k=0}^{n-1}\frac{1}{n+a_k} \geq \sum_{k=0}^{n-1}\frac{1}{n+a_n} = \frac{n}{n+a_n} \geq \frac{n}{n+\frac{2n+1}{2n+2}} = \frac{2n(n+1)}{2n^2+4n+1}$.

Hence, $\frac{1}{a_n} \leq 2 - \frac{2n(n+1)}{2n^2+4n+1} = \frac{2n^2+6n+2}{2n^2+4n+1} \Leftrightarrow$

$$a_n \geq \frac{2n^2+4n+1}{2n^2+6n+2} = 1 - \frac{2n+1}{2n^2+6n+2}$$

and $a_n - \left(1 - \frac{1}{n+2}\right) \geq 1 - \frac{2n+1}{2n^2+6n+2} - \left(1 - \frac{1}{n+2}\right) = \frac{n}{2(n+2)(n^2+3n+1)} > 0$.